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Countability and star covering properties ☆

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Whenever P is a topological property, we say that a topological space is *star* P if whenever $\mathcal{U}$ is an open cover of X, there is a subspace $A \subseteq X$ with property P such that $X = \mathrm{St}(A, \mathcal{U})$. We study the relationships of star P properties for $P \in \{\text{Lindelöf}, \ \sigma\text{-compact}, \text{countable}\}$ with other Lindelöf type properties.

1. Introduction and preliminary results

If X is a topological space and $\mathcal{U}$ is a family of subsets of X, then the *star* of a subset $A \subseteq X$ with respect to $\mathcal{U}$ is the set $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon\ U \cap A \neq \emptyset\}$.

Definition 1.1. Let P be a topological property. A space X is said to be *star* P if whenever $\mathcal{U}$ is an open cover of X, there is a subspace $A \subseteq X$ with property P such that $X = \mathrm{St}(A, \mathcal{U})$. The set A will be called a *star kernel* of the cover $\mathcal{U}$.

The term star P was coined in [11] but certain star properties, specifically those properties corresponding to "$\mathcal{P} =$ *finite*" and "$\mathcal{P} =$ *countable*" were first studied by van Douwen et al. in [1] and later by many other authors. A survey of star properties with a comprehensive bibliography can be found in [12]. We believe our terminology to be simple and logical, but we must mention that authors of previous works have used many different notations to define properties of this sort. For example, in [12] and earlier in [1], the terms *starcompact* and *strongly* 1-*starcompact* have been used in place of the concept we would here call star finite. The term *strongly* 1-*star Lindelöf* used in previous papers is equivalent to the property we

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here call star countable. Furthermore, the property called star Lindelöf in this paper has been briefly studied in [4] and [5] under the rather confusing name Lindelöf-starcompact.

In this paper we shall be concerned with properties P related to the Lindelöf covering property, specifically, "$P =$ countable", "$P =$ Lindelöf" and "$P = \sigma$-compact". In the remainder of this section, spaces are assumed only to be T_1.

Every space is star discrete; in fact, given any open cover $\mathcal{U}$ of a space X, there is a star kernel of $\mathcal{U}$ which is a closed and discrete subspace of X. It follows immediately, that if a space is not star countable, then it has an uncountable closed and discrete subspace. Thus we have the following well-known trivial result.

Theorem 1.2. *A space of countable extent is star countable.*

The next result also has a trivial proof.

Theorem 1.3. *If P is a property preserved under continuous images then the property star P is also preserved under continuous images.*

Thus a continuous image of a star countable (respectively star σ-compact, star Lindelöf) space is star countable (respectively star σ-compact, star Lindelöf).

The construction contained in the following remark will be useful in the sequel for studying products.

Remark 1.4. If K is compact and $\mathcal{U}$ is an open cover of $X \times K$ by basic open sets, then for each $x \in X$, there is an open set W_x in X such that $W_x \times K$ is covered by a finite number of elements of $\mathcal{U}$, say

$$W_x \times K \subseteq \bigcup \bigl\{U_k(x) \times V_k(x)\colon\ 1 \leqslant k \leqslant n_x\bigr\}$$

and where $W_x \subseteq \bigcap\{U_k(x)\colon\ 1 \leqslant k \leqslant n_x\}$.

Definition 1.5. We say that a topological property P is *compactly productive* if whenever X has P and Y is compact, then $X \times Y$ has P.

Theorem 1.6. *If P is a compactly productive property, then so is star P.*

Proof. Suppose that X is star P and K is compact. Let $\mathcal{U}$ be a basic open cover of $X \times K$. Using the notation of Remark 1.4, let $\mathcal{W} = \{W_x\colon\ x \in X\}$. Since X is star P, there is a subspace $A \subseteq X$ with property P such that $\mathrm{St}(A, \mathcal{W}) = X$. Then $A \times K$ has property P and $\mathrm{St}(A \times K, \mathcal{U}) = X \times K$. □

Corollary 1.7. *([4]) The product of a star Lindelöf space and a compact space is star Lindelöf.*

Corollary 1.8. *The product of a star σ-compact space and a compact space is star σ-compact.*

Countability is clearly not a compactly productive property. Example 3.3.4 of [1] illustrates the fact that the product of an arbitrary compact (Hausdorff) space with a star countable space need not be star countable. We will discuss this example later in the next section. Nonetheless, we have the following result, mentioned prior to 3.3.4 in [1].

Theorem 1.9. *The product of a star countable space and a separable compact space is star countable.*

The proof of the this result is similar to that of Theorem 1.6, but now there is a countable subspace $A \subseteq X$ such that $\mathrm{St}(A, \mathcal{W}) = X$ and there is a countable dense subspace $D \subseteq K$. Then $\mathrm{St}(A \times D, \mathcal{U}) = X \times K$.

2. Star Lindelöf spaces

As mentioned previously, star countable spaces have been studied by many authors (under the name star Lindelöf). The focus of this section will be a study of the property star Lindelöf (and other related properties) as we have defined it in the Introduction: A space is *star Lindelöf* if every open cover possesses a Lindelöf star kernel. Many of the results presented, particularly those of Section 1 above are valid in the class of T_1-spaces. In this and succeeding sections, all spaces are T_3 unless stated otherwise. Undefined terms can be found in [2] or [7]. We start with statements of some easy consequences of the definitions, the first result is well known.

Lemma 2.1. *Both separable and Lindelöf spaces are star countable.*

Lemma 2.2. *Star countable $\Rightarrow$ star σ-compact $\Rightarrow$ star Lindelöf.*

Lemma 2.3. *A space is star countable if and only if it is star separable.*

Proof. The necessity is obvious. For the sufficiency, suppose that X is star separable and $\mathcal{U}$ is an open cover of X. Let Z be a separable subspace of X such that $X = \mathrm{St}(Z, \mathcal{U})$ and let D be a countable dense subspace of Z. If $x \in X$, then there is some $z \in Z$ such that $x \in \mathrm{St}(z, \mathcal{U})$ and thus there is some $U \in \mathcal{U}$ such that $x, z \in U$. Since D is dense in Z, there is some $d \in D \cap U$ and so $x \in \mathrm{St}(D, \mathcal{U})$. $\square$

The property of being star Lindelöf (as defined here) is not equivalent to being star countable as the following example shows. (A comment on p. 98 of [12], states that such an example also appears in [5].)

Example 2.4. There is a Tychonoff space which is star Lindelöf but not star σ-compact.

Proof. Let Y be the ω-modification of $\omega_1 + 1$ (that is to say, the topology generated by the G_δ sets of the order topology), $X = \omega_2 + 1$ with the order topology and $Z = (X \times Y) \setminus \{(\omega_2, \omega_1)\}$. Note that a compact subspace of Y is necessarily finite. To show that Z is star Lindelöf, suppose that $\mathcal{V}$ is an open cover of Z. For each $\alpha \in \omega_1 \subseteq Y$, we can pick $V_\alpha \in \mathcal{V}$ so that $(\omega_2, \alpha) \in V_\alpha$ and $\beta_\alpha \in \omega_2$ so that $]\beta_\alpha, \omega_2] \times \{\alpha\} \subseteq V_\alpha$. Let $\beta = \sup\{\beta_\alpha\colon \alpha \in \omega_1\}$ and $A = \{\beta + 1\} \times Y$; then $S =]\beta + 1, \omega_2] \times \omega_1 \subseteq \mathrm{St}(A, \mathcal{V})$. However, $Z \setminus S$ is the union of the subspaces $[0, \beta + 1] \times Y$ and $\omega_2 \times \{\omega_1\}$ and the former is Lindelöf while the latter is countably compact. Thus there is a countable set $C \subseteq Z$ and a finite set $F \subseteq Z$ such that $X \setminus S \subseteq \mathrm{St}(F, \mathcal{V}) \cup \mathrm{St}(C, \mathcal{V})$ and so $A \cup C \cup F$ is the required Lindelöf subspace of Z.

To see that Z is not star σ-compact, it suffices to consider the open cover $\mathcal{U} = \{\omega_2 \times Y\} \cup \{X \times \{\alpha\}\colon \alpha \in \omega_1\}$. To complete the proof note that every compact subset $C \subseteq Z$ is contained in $X \times F$, where F is a finite subset of Y, for otherwise $\pi_Y(C)$ is infinite and so is not compact. Thus every σ-compact subspace of Z is contained in $X \times L$, where L is a countable subset of Y. $\square$

Example 3.3.4 of [1] is an example of a space which is star σ-compact but not star countable. For completeness we define the space.

Example 2.5. There is a Tychonoff space which is star σ-compact but not star countable.

Proof. Let $\Psi = \omega \cup \mathcal{A}$ be a Mrowka space of, say, cardinality $\mathfrak{c}$ (where $\mathcal{A}$ is a MAD family on ω), Y be the one-point compactification of the discrete space D of size $\mathfrak{c}$ and $X = \Psi \times Y$. Since Ψ is separable it is star countable and hence star σ-compact. Since star σ-compactness is a compactly productive property, X is star σ-compact. If $\{A_\alpha\colon \alpha \in \mathfrak{c}\}$ and $\{d_\alpha\colon \alpha \in \mathfrak{c}\}$ are enumerations of $\mathcal{A}$ and D respectively, then the open cover

$$\mathcal{U} = \{\Psi \times \{d_\alpha\}\colon \alpha < \mathfrak{c}\} \cup \{(A_\alpha \cup \{A_\alpha\}) \times (Y \setminus \{d_\alpha\})\colon \alpha < \mathfrak{c}\} \cup \{\{n\} \times Y\colon n \in \omega\},$$

witnesses the fact that X is not star countable. $\square$

Note that Example 2.4 has uncountable cellularity and contains an uncountable closed discrete subset. However, as Theorem 2.7 (below) shows, a star Lindelöf space may not possess an uncountable discrete family of open sets. It is also clear from this example that a closed subspace of a star Lindelöf space need not be star Lindelöf, although it is an easy exercise to show that the property is inherited by regular closed subspaces.

Theorem 2.6. *Every Tychonoff space can be embedded as a closed G_δ in a Tychonoff star σ-compact space.*

Proof. Let $Y = (\beta X \times \omega) \cup (X \times \{\omega\})$, where Y has the relative topology inherited from $\beta X \times (\omega + 1)$. Note that $\beta X \times \omega$ is σ-compact and dense in Y and hence is a σ-compact star kernel for each open cover of Y. $\square$

Theorem 2.7. *If X is a star Lindelöf space, then every locally finite family of non-empty open sets in X is countable.*

Proof. Suppose to the contrary that $\mathcal{F} = \{F_\alpha\colon \alpha \in I\}$ is an uncountable locally finite family of non-empty open sets. For each $\alpha \in I$, we pick $x_\alpha \in F_\alpha$. Then $A = \{x_\alpha\colon \alpha \in I\}$ is closed and if we let $\mathcal{V} = \{X \setminus \{x_\alpha\colon \alpha \in I\}\} \cup \{F_\alpha\colon \alpha \in I\}$, then $\mathcal{V}$ is an open cover of X which has no countable kernel since each set F_α contains at most finitely many points of A. $\square$

Corollary 2.8. *A space which admits an uncountable partition into open sets is not star Lindelöf.*

Both countably compact and Lindelöf spaces are star Lindelöf. The next result shows that, as is to be expected, the star Lindelöf property is not preserved under arbitrary finite products.

Corollary 2.9. *The product of a Lindelöf space and a countably compact space need not be star Lindelöf.*

Proof. Again we let Y denote the ω-modification of ω_1+1 and $X=\omega_1\times Y$. Then $\mathcal{P}=\{U_\alpha\colon \alpha\in\omega_1\}\cup\{V\}$, where $V=\{(\alpha,\beta)\colon \alpha\in\omega_1,\ \beta\in\omega_1+1,\ \beta\geqslant\alpha\}$ and for each $\alpha\in\omega_1$, $U_\alpha=]\alpha,\omega_1[\times\{\alpha\}$, is an uncountable partition of X into open sets. □

Recall that a space is *ω_1-Lindelöf* if every open cover of size ω_1 has a countable subcover or equivalently, if every subset of size ω_1 has a complete accumulation point. Following [12] we say that a space X is *feebly Lindelöf* if every locally finite family of non-empty open sets in X is countable. Theorem 2.7 states that a star Lindelöf space is feebly Lindelöf. Using terminology from [14], say that a space has the *DCCC property* if each discrete family of non-empty open subsets is countable. Obviously a feebly Lindelöf space has the *DCCC* property and the two properties are equivalent in the class of regular (even weakly regular) spaces.

The proof that the ω_1-Lindelöf property is compactly productive is similar to the corresponding proof for Lindelöf spaces.

Theorem 2.10. *If X is a feebly Lindelöf space and Y is separable, then $X\times Y$ is feebly Lindelöf.*

Proof. Suppose to the contrary that $X\times Y$ is not feebly Lindelöf and that $\mathcal{C}$ is an uncountable locally finite family of open sets in $X\times Y$. We assume without loss of generality that $|\mathcal{C}|=\omega_1$ and that each element of $\mathcal{C}$ is a basic open set in the product, that is

$$\mathcal{C}=\{U_\alpha\times V_\alpha,\ \alpha\in\omega_1\},$$

where U_α is open in X and V_α is open in Y. Let D be a countable dense subset of Y; for each $d\in D$, the set $\{U_\alpha\colon \alpha\in\omega_1 \text{ and } d\in V_\alpha\}$ is locally finite in X and hence is countable. It follows that $\{\alpha\in\omega_1\colon d\in V_\alpha\}$ is countable and so $\omega_1=\bigcup\{\{\alpha\in\omega_1\colon d\in V_\alpha\}\colon d\in D\}$ is also countable, a contradiction. □

The following implications are now clear:

$$\begin{aligned}&\omega_1\text{-Lindelöf} \quad\Rightarrow\quad \text{countable extent} \quad\Rightarrow\quad \text{star countable}\\ &\quad\Rightarrow\quad \text{star } \sigma\text{-compact} \quad\Rightarrow\quad \text{star Lindelöf} \quad\Rightarrow\quad \text{feebly Lindelöf.}\end{aligned}$$

Examples to show that none of these implications can be reversed are as follows:

(1) The space ω_1 with the order topology has countable extent but is not ω_1-Lindelöf.
(2) A Mrowka space Ψ, is separable, hence star countable, but has uncountable extent.
(3) Example 2.5 is a space which is star σ-compact but not star countable.
(4) Example 2.4 above is a space which is star Lindelöf but not star σ-compact.
(5) Consider the subspace $X=(\omega_1\times\omega)\cup(S\times\{\omega\})$ where S is the set of all isolated points of ω_1. Note that $\omega_1\times\omega$ is an open dense subspace of X which is a countable union of pseudocompact spaces and hence is feebly Lindelöf; it follows immediately that X is feebly Lindelöf. On the other hand, the open cover $\mathcal{U}=\{\{\alpha\}\times(\omega+1)\colon \alpha\in S\}\cup\{\omega_1\times\omega\}$ witnesses that X is not star Lindelöf since any star kernel of $\mathcal{U}$ must contain points with arbitrarily large first coordinate.

We wish to thank the anonymous referee for providing us with this last example and also for a considerable simplification of Example 3.3 in the next section.

3. When does a feebly Lindelöf space have countable extent?

The first two theorems describe classes for which the answer to the question in the title of this section is affirmative.

Recall that a space is *strongly collectionwise* (respectively, *collectionwise*) *Hausdorff* if every discrete family of points can be separated by a discrete (respectively, pairwise disjoint) family of open sets. It is immediate from the definition of a feebly Lindelöf space that a strongly collectionwise Hausdorff space has the *DCCC* property if and only if it has countable extent. As an immediate consequence, we have the following result.

Theorem 3.1. *A GO-space is feebly Lindelöf* (*or has the DCCC property*) *if and only if it has countable extent.*

A space X is a *P-space* if each G_δ subset of X is open. It is easy to see that a completely regular P-space is zero-dimensional.

Theorem 3.2. *A normal P-space is feebly Lindelöf if and only if it has countable extent.*

Proof. Suppose to the contrary that $D = \{d_\alpha\colon\ \alpha \in \omega_1\}$ is a closed discrete subspace of cardinality ω_1. Pick an open and closed set U_0 such that $U_0 \cap D = \{d_0\}$. Having chosen disjoint clopen sets U_α for each $\alpha < \beta \in \omega_1$, note that $\bigcup\{U_\alpha\colon\ \alpha < \beta\}$ is closed and we may find an open set U_β such that $U_\beta \cap D = \{d_\beta\}$ and $U_\beta \cap U_\alpha = \emptyset$ for each $\alpha < \beta$. Thus $\mathcal{U} = \{U_\alpha\colon\ \alpha \in \omega_1\}$ is a family of mutually disjoint clopen sets. The closed sets $X \setminus \bigcup \mathcal{U}$ and D are disjoint and hence there exist disjoint open sets U and V such that $D \subseteq U$ and $X \setminus \bigcup \mathcal{U} \subseteq V$. It is clear that the family $\{U_\alpha \cap U\colon\ \alpha \in \omega_1\}$ is an uncountable locally finite family of open sets in X. $\square$

A more subtle question is whether every feebly Lindelöf P-space is star Lindelöf. With regard to this question, the following consistent example would appear interesting. First, recall from [10] that if D is an infinite set, a family $\mathcal{F} \subseteq \mathcal{P}(D)$ is *almost disjoint* if for each pair of distinct sets $F, G \in \mathcal{F}$, $|F| = |G| = |D|$ but $|F \cap G| < |D|$. It is a simple consequence of Zorn's Lemma that each such family is contained in a family which is maximal with respect to being almost disjoint (a MAD family) and it is well known that if $\mathcal{F}$ is an infinite MAD family on a set of regular cardinality κ, then $|\mathcal{F}| > \kappa$. If $\mathcal{A} \subseteq [\omega_1]^{\omega_1}$ is a MAD family of subsets of ω_1, such that $\bigcup \mathcal{A} = \omega_1$, of cardinality κ then we may define a topology τ on $X = \omega_1 \cup \mathcal{A}$ by declaring each $d \in \omega_1$ to be isolated and a neighbourhood of $A \in \mathcal{A}$ is of the form $\{A\} \cup (A \setminus C)$, where C is countable. The space (X, τ) is called a *Mrowka space on* ω_1 and since $\mathcal{A}$ is MAD, it is a simple exercise to show that (X, τ) is feebly Lindelöf.

Example 3.3. If there exists a MAD family $\mathcal{A}$ on ω_1 such that $|\mathcal{A}|^\omega = |\mathcal{A}| = \kappa$, then there is a feebly Lindelöf P-space which is not star Lindelöf.

Proof. Let $X = \omega_1 \cup \mathcal{A}$ be a Mrowka space on ω_1. It is clear that X is a feebly Lindelöf P-space and we will show that under the set theoretic hypothesis $|\mathcal{A}|^\omega = |\mathcal{A}|$, X is not star Lindelöf.

First note that since $\bigcup \mathcal{A} = \omega_1$, $\{A \cup \{A\}\colon\ A \in \mathcal{A}\}$ is an open cover of X and hence every Lindelöf subspace of X has a countable cover of the form $\{A \cup \{A\}\colon\ A \in \mathcal{F}\}$, for some countable $\mathcal{F} \subseteq \mathcal{A}$: Furthermore each of the sets $\bigcup\{F \cup \{F\}\colon\ F \in \mathcal{F}\}$ is Lindelöf and is closed in X, being the countable union of closed subspaces of a P-space. It follows that if $L \subseteq X$ is Lindelöf, then there is a countable subset $\mathcal{F} \subseteq \mathcal{A}$ such that $L \subseteq \bigcup\{F \cup \{F\}\colon\ F \in \mathcal{F}\}$ and clearly there are $\kappa^\omega = \kappa$ such subspaces of this type in X.

Now fix a bijection $\phi : \mathcal{A} \to [\mathcal{A}]^\omega$ in such a way that for each $A \in \mathcal{A}$, $A \notin \phi(A)$. To do this, fix enumerations $\{A_\alpha\colon\ \alpha \in \kappa\}$ of $\mathcal{A}$ and $\{\mathcal{G}_\alpha\colon\ \alpha \in \kappa\}$ of $[\mathcal{A}]^\omega$, and for each A_α define $\phi(A_\alpha)$ recursively to be $\mathcal{G}_{\beta_\alpha}$, where $\beta_\alpha \in \kappa$ is minimal so that $A_\alpha \notin \mathcal{G}_{\beta_\alpha}$ and $\mathcal{G}_{\beta_\alpha} \neq \mathcal{G}_{\beta_\gamma}$ for each $\gamma < \alpha$. It is easy to see that this choice is possible since for each $\alpha \in \kappa$, $\{\beta \in \kappa\colon\ A_\alpha \notin \mathcal{G}_\beta\}$ has cardinality κ. For each $\alpha \in \kappa$ define

$$U_\alpha = \{A_\alpha\} \cup \Big(A_\alpha \setminus \bigcup\big\{B \cap A_\alpha\colon\ B \in \phi(A_\alpha)\big\}\Big).$$

Since $\mathcal{A}$ is an almost disjoint family, each U_α is open and we let

$$\mathcal{U} = \{U_\alpha\colon\ \alpha \in \kappa\} \cup \big\{\{d\}\colon\ d \in \omega_1\big\}.$$

We claim that $\mathcal{U}$ witnesses the fact that X is not star Lindelöf. To prove our claim, suppose to the contrary that there is a Lindelöf subspace L of X such that $\mathrm{St}(L, \mathcal{U}) = X$. Then as we have shown, there is some $\mathcal{F} \in [\mathcal{A}]^\omega$ such that $L \subseteq \{A \cup \{A\}\colon\ A \in \mathcal{F}\}$ and there is some $A_\xi \in \mathcal{A}$ such that $\phi(A_\xi) = \mathcal{F}$. However, $A_\xi \in \mathrm{St}(L, \mathcal{U})$ and so there is $d \in L$ and $U \in \mathcal{U}$ such that $d, A_\xi \in U$. Necessarily, $U = U_\xi = \{A_\xi\} \cup (A_\xi \setminus \bigcup\{B \cap A_\xi\colon\ B \in \phi(A_\xi)\})$; but then, $d \in A_\xi \setminus \bigcup\{B \cap A_\xi\colon\ B \in \phi(A_\xi)\}$ and $d \in \bigcup \phi(A_\xi)$, a contradiction. $\square$

The existence of a MAD family $\mathcal{A}$ on ω_1 such that $|\mathcal{A}|^\omega = |\mathcal{A}|$ is independent of *ZFC*: Exercise (B5) of Chapter 8 of [10], postulates the existence of a model of *ZFC* in which $2^\omega = 2^{\omega_1} = 2^{\omega_2} = \omega_3$ and there is no almost disjoint family on ω_1 of size ω_3. Thus in this model, there are ${\omega_1}^\omega = \omega_3$ Lindelöf subspaces but $|\mathcal{A}| = \omega_2$. However, the condition $|\mathcal{A}|^\omega = |\mathcal{A}|$ is a consequence of $2^{\omega_1} = \omega_2$ (since then necessarily $|\mathcal{A}| = 2^{\omega_1}$) and also by Theorem 1.3 of Chapter 2 of [10] it is a consequence of *CH*. However, we ask:

Problem 3.4. Is it consistent with *ZFC* that the space described above is star Lindelöf?

Problem 3.5. Is it consistently true that a feebly Lindelöf P-space is star Lindelöf?

Under $MA + \neg CH$ an ω_1-Cantor tree is an example of a normal feebly Lindelöf space which has uncountable extent. Contrarily, it was shown in [3], that under $V = L$, a normal space of character at most ω_1 is strongly collectionwise Hausdorff and so as an immediate corollary, a first countable, normal, feebly Lindelöf space has countable extent. The next theorem generalizes this result, using a much weaker set-theoretic hypothesis; the proof is rather similar to that of Theorem 2.2 of [13].

Theorem 3.6. *($2^\omega < 2^{\omega_1}$) If X is a normal, feebly Lindelöf space and $\chi(X) \leqslant \mathfrak{c}$, then X has countable extent.*

Proof. Suppose to the contrary that X has the properties given in the hypothesis, but X has uncountable extent; let F be a closed discrete subset of X of cardinality ω_1. For each $x \in F$, let $\mathcal{B}_x$ be a local base at x of cardinality at most $\mathfrak{c}$. Since X is normal, for each non-empty set $G \subseteq F$ fix an open set U_G such that $G \subseteq U_G$ and $\mathrm{cl}(U_G) \cap (F \setminus G) = \emptyset$. For each such G, choose a maximal pairwise disjoint collection of members of $\bigcup\{\mathcal{B}_x\colon x \in G\}$ which are contained in U_G; denote this family by $\mathcal{A}_G$. Then $G \subseteq \mathrm{cl}(\bigcup \mathcal{A}_G)$ and if $G \neq H$, then $\mathcal{A}_G \neq \mathcal{A}_H$. We consider two cases:

(1) Each family $\mathcal{A}_G$ is countable; or
(2) There is some uncountable $\mathcal{A}_G$.

If (1) occurs, then since $|F| = \omega_1$ and for each $x \in F$, $|\mathcal{B}_x| \leqslant \mathfrak{c}$, it follows that $|\bigcup\{\mathcal{B}_x\colon x \in F\}| \leqslant \mathfrak{c}$ and $|\{\mathcal{A}_G\colon \emptyset \subsetneq G \subsetneq F\}| \leqslant \mathfrak{c}$. However, the function $G \mapsto \mathcal{A}_G$ is injective and so $|\{\mathcal{A}_G\colon \emptyset \subsetneq G \subsetneq F\}| = 2^{\omega_1}$, a contradiction.

If (2) occurs, then we can pick $\emptyset \subsetneq G \subsetneq F$ so that $\mathcal{A}_G$ is uncountable and hence find an uncountable set $A \subseteq G$ which is separated by the elements of $\mathcal{A}_G$. Since X is normal, there is a locally finite collection of open sets which is uncountable, again a contradiction. □

Corollary 3.7. *Under CH, a normal first countable feebly Lindelöf space has countable extent.*

The cellularity of a space Y is denoted by $c(Y)$. For a space X we define

$$hLc(X) = \sup\bigl\{c(Y)\colon Y \text{ is a Lindelöf subspace of } X\bigr\}.$$

Recall that a space X is *weakly collectionwise Hausdorff* if whenever F is a closed discrete subspace of X, there is a subset $A \subseteq F$ such that $|A| = |F|$ and the points of A can be separated by a pairwise disjoint family of open sets.

Theorem 3.8. *Let X be a star Lindelöf space. Then*:

(1) *If X is weakly collectionwise Hausdorff then $e(X) \leqslant hLc(X)$; and*
(2) *If X is Hausdorff, then $e(X) \leqslant exp(\chi(X) \cdot hLc(X))$.*

Proof. (1) Suppose that $e(X) > \kappa$ and D is a closed and discrete subset of X, $|D| = \kappa^+$. There is a subset $A \subseteq D$, of cardinality κ^+ and the points of A can be separated by a pairwise disjoint family $\mathcal{F}$ of open sets. Then $\mathcal{F} \cup \{X \setminus A\}$ is an open covering of X which must have a Lindelöf star kernel L. Clearly L meets each element of $\mathcal{F}$ and so $c(L) \geqslant |A| = \kappa^+$ showing that $hLc(X) \geqslant \kappa^+$.

(2) Let $\kappa = \chi(X) \cdot hLc(X)$; for each $x \in X$, enumerate (not necessarily faithfully) a local base $\{V_\alpha(x)\colon \alpha < \kappa\}$ at x. Suppose to the contrary that $e(X) > 2^\kappa$ and let C be a closed and discrete subset of X of cardinality $(2^\kappa)^+$. Fix a well-order $<$ on C and for each distinct $x, y \in C$, pick $\alpha(x, y), \beta(x, y) \in \kappa$ so that $V_{\alpha(x,y)}(x) \cap V_{\beta(x,y)}(y) = \emptyset$; now define $\phi : [C]^2 \to \kappa \times \kappa$ by $\phi(\{x, y\}) = (\alpha(x, y), \beta(x, y))$ when $x < y$. Since $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$, there is $A \subseteq C$ of size κ^+ such that ϕ is constant on $[A]^2$, say $\phi(\{x, y\}) = (\alpha_0, \beta_0)$ for all $\{x, y\} \in [A]^2$ (see 0.4 of [7] and Chapter 9 of [6]). Let $U_x = V_{\alpha_0}(x) \cap V_{\beta_0}(x)$; the κ^+-many sets $\{U_x\colon x \in A\}$ are pairwise disjoint and the proof follows exactly as in (1). □

A Mrowka space Ψ of cardinality $\mathfrak{c}$ is a first countable, star Lindelöf T_3-space in which every Lindelöf subspace is countable and $e(\Psi) = \mathfrak{c}$. Thus the inequality in (2) is the best possible.

4. Feebly Lindelöf subspaces of $\omega_1 \times \omega_1$

The question remains as to when a feebly Lindelöf (or *DCCC*) space has countable extent. We have seen that this is the case in the class of GO-spaces, in the class of normal P-spaces and, under $V = L$, for normal first countable spaces. On the other hand as mentioned above, under $MA + \neg CH$, an ω_1-Cantor tree is a normal first countable feebly Lindelöf space which has uncountable extent. We proceed to show that feebly Lindelöf subproducts of ${\omega_1}^2$ have countable extent. We denote the family of stationary subsets of ω_1 by $\mathcal{S}(\omega_1)$. We need the following result:

Theorem 4.1. *([8]) If $A, B \in \mathcal{S}(\omega_1)$ then the following are equivalent*:

(1) *$A \cap B \in \mathcal{S}(\omega_1)$.*
(2) *$A \times B$ is normal.*
(3) *Every uncountable subset of $A \times B$ has an accumulation point.*

Theorem 4.2. *If $A, B \in \mathcal{S}(\omega_1)$, then the following conditions are equivalent*:

(i) *$A \times B$ is feebly Lindelöf.*

(ii) *$A \times B$ has countable extent.*
(iii) *$A \times B$ is normal.*

Proof. It is clear that (ii) $\Rightarrow$ (i) and the proof of the equivalence of (ii) and (iii) follows from Theorem 4.1. Thus we need only prove that (i) $\Rightarrow$ (ii). To this end, suppose that $A, B \in \mathcal{S}(\omega_1)$, but $A \cap B \notin \mathcal{S}(\omega_1)$. By a result of [8], there is a closed discrete set $D \subseteq A \times B$ of cardinality ω_1, but for our purposes here we give a direct construction of such a set.

Let C be a club in ω_1 such that $C \cap A \cap B = \emptyset$ and enumerate C as $\{\gamma_\alpha\colon \alpha \in \omega_1\}$. Now define

$$\alpha_0 = \min(C \cap A),$$
$$\beta_0 = \min\big((C \cap B)\backslash(\alpha_0 + 1)\big),$$
$$\vdots$$
$$\alpha_\delta = \min\big((C \cap A)\backslash\big(\sup\{\beta_\mu\colon \mu < \delta\} + 1\big)\big),$$
$$\beta_\delta = \min\big((C \cap B)\backslash\big(\sup\{\alpha_\mu\colon \mu \leqslant \delta\} + 1\big)\big).$$

The set $D = \{(\alpha_\delta, \beta_\delta)\colon \delta < \omega_1\}$ thus constructed recursively is closed and discrete, for if $\langle(\alpha_{\delta_n}, \beta_{\delta_n})\rangle \to (\alpha, \beta)$, then necessarily $\alpha = \beta$, which implies that $\alpha \in A \cap B \cap C$, a contradiction. Now we consider points of the form $(\alpha_{\delta+1}, \beta_{\delta+1}) \in D$. If $(\alpha_{\delta+1}, \beta_{\delta+1})$ is isolated in $A \times B$, then define $(a_{\delta+1}, b_{\delta+1}) = (\alpha_{\delta+1}, \beta_{\delta+1})$; otherwise, since $A \times B$ is dispersed, we can pick an isolated point

$$(a_{\delta+1}, b_{\delta+1}) \in V_{\delta+1} = \big(]\alpha_\delta, \alpha_{\delta+1}] \times]\beta_\delta, \beta_{\delta+1}]\big) \cap (A \times B).$$

Note that $\alpha_\delta < a_{\delta+1} \leqslant \alpha_{\delta+1}$ and $\beta_\delta < b_{\delta+1} \leqslant \beta_{\delta+1}$. These inequalities imply that any accumulation point of $\{(a_{\delta+1}, b_{\delta+1})\colon \delta \in \omega_1\}$ would also be an accumulation point of $\{(\alpha_\delta, \beta_\delta)\colon \delta < \omega_1\}$ which is closed and discrete. Thus the discrete uncountable family of open sets $\{\{(a_\delta, b_\delta)\}\colon \delta < \omega_1\}$ witnesses that $A \times B$ is not feebly Lindelöf. $\square$

Theorem 4.3. *A subspace of ${\omega_1}^2$ is star Lindelöf if and only if it has countable extent* (*and hence is star countable*).

Proof. Assume to the contrary that $X \subseteq {\omega_1}^2$ is star Lindelöf but has uncountable extent; let $F \subseteq X$ be a closed, discrete, subset of X of cardinality ω_1. Since X is collectionwise Hausdorff (see [9]), there is a pairwise disjoint family of open sets $\mathcal{F} = \{U_x\colon x \in F\}$ such that $x \in U_x$ for each $x \in F$. Consider the open cover $\mathcal{F} \cup \{X \backslash F\}$; since X is star Lindelöf, there is a Lindelöf subspace $A \subseteq X$ such that $\mathrm{St}(A, \mathcal{U}) = X$. For each $x \in F$, there is some $a \in A$ and $U \in \mathcal{U}$ such that $a, x \in V$. However, this implies that $V = U_x$ and therefore $a \in U_x$, showing that $\{U_x \cap A\colon x \in F\}$ is an uncountable cellular family in A. This is a contradiction, since every Lindelöf subspace of ${\omega_1}^2$ is countable. $\square$

5. Open questions

In addition to the problems posed in Section 3, the following questions might be interesting.

(1) Is a first countable feebly compact space star Lindelöf?
(2) Is a first countable star Lindelöf space, star countable?
(3) Is a pseudocompact Tychonoff space star Lindelöf?
(4) Is a T_4 feebly Lindelöf space star Lindelöf? Is a T_4 star Lindelöf space star countable?
(5) Are the classes of spaces studied here closed under perfect (closed, open) preimages?

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